Integration: A General Procedure

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Abstract: An effective strategy is here presented for solving problems in integration, targeted to first year calculus students already familiar with basic notions of calculus and anti-derivatives.

1 Introduction

If a function f is continuous over the interval $[a,b]$, then the function g defined as in (1.1) below, is continuous over the same interval, differentiable on (a,b), and $g'(x) = f(x)$ by the fundamental theorem of calculus.

$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b \tag{1.1}$$

One can therefore easily integrate basic elementary functions by finding or using known anti-derivatives. In numerous instances however, this is insufficient, and additional procedures are required.

Standard methods can be followed to evaluate an integral. Some of these tactics will be presented in this paper, including u-substitution, integration by reduction and partial fractions, and trigonometric substitution. An appendix is provided for quick referencing of common derivatives and integral forms.

NOTICE:

Before beginning complex and time-consuming procedures, it is often worth checking for easier alternative routes, such as using algebraic manipulations or substitution by known identities, to simplify the integral We give such an example in (1.2) below.

$$\begin{aligned}\int(\sin x+\cos x)^2 dx &= \int(\sin^2 x+2\sin x\cos x+\cos^2 x)dx \\ &= \int 1dx+\int \sin(2x)dx \\ &= x-\frac{1}{2}\cos(2x)+C\end{aligned} \tag{1.2}$$

2 U-Substitution

Integrals of composition or products of functions can often be evaluated through a technique named u-substitution. The underlying principle of this modus operandi relies on the substitution rule for definite (2.1) and indefinite (2.2) integrals.

Let u = g(x), g a differentiable function with a range of $[a, b]$, and f a function continuous on $[a, b]$. Then:

$$\int f(g(x))g^{'}(x)dx = \int f(u)du \qquad (2.1)$$

$$\int_a^b f(g(x))g^{'}(x)dx = \int_{g(a)}^{g(b)} f(u)du \qquad (2.2)$$

It is important to note that dx and du can be operated after an integral as if they were differentials.

U-substitution is used most often to replace functions that are a parameter of another function, denominators, or generally vicious terms. (2.3) is such an example, where an "inner" function is substituted. Note that generally for u-substitution to work, the derivative of u must be able to "absorb" and cancel out the other non substituted variable.

Given $\int x^4(1+x^5)^3 dx$, we wish to substitute the "inner" function, that is, set $u = (1+x^5)$. Therefore: $du = 5x^4 dx$

$$\begin{aligned}\int x^4(1+x^5)^3 dx &= \frac{1}{5}\int (1+x^5)^3(5x^4)dx \qquad (2.3)\\ &= \frac{1}{5}\int u^3 du\\ &= \left(\frac{1}{5}\right)\left(\frac{1}{4}\right)u^4\\ &= \left(\frac{1}{20}\right)(1+x^5)^4\end{aligned}$$

3 Integration by reduction

If f and g are differentiable functions, then through the product rule for differentiation, we obtain

$$\int f(x)g^{'}(x)dx = f(x)g(x) - \int f^{'}(x)g(x)dx \qquad (3.1)$$

Using the substitution rule, we get

$$\int u dv = uv - \int v du \qquad (3.2)$$

When choosing u and dv, you must be sure that you are able to integrate dv, and that the term $\int v du$ is easier to compute than the original integral. This often occurs when u is simplified by differentiation. Integration by reduction is often used when the integrand is a product of disparate functions, as in example (3.3) below.

Given the integrand $\int x \cos x dx$, let $u = x$, and $dv = \cos x dx$. By using formula (3.2) we then obtain

$$\begin{aligned} \int (x)(\cos x dx) &= (x)(\sin x) - \int (\sin x)(dx) \qquad (3.3) \\ &= x \sin x + \cos x + C \end{aligned}$$

4 Trigonometric substitution

By using the substitution rule backwards, and given a one-to-one function $g(t)$, we can define inverse substitution as the following:

$$\int f(x) dx = \int f(g(t)) g^{'}(t) dt \qquad (4.1)$$

In this situation, trigonometric functions can be used if we restrict their domain such that they remain one-to-one functions. Combined with trigonometric identities (Appendix 2), this process can be an invaluable tool.

In the expression $\sqrt{a^2 - x^2}$, by using the substitution $x = a \sin t$ over a restricted domain $\left[-\frac{p}{2}, \frac{p}{2}\right]$, and the trigonometric identity $1 - \sin^2 x = \cos^2 x$, we can simplify the term to $a|\cos t|$. However, considering the constrained domain, we can further simplify the expression to $a \cos t$.

Similarly, we can simplify the terms $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$, by using the substitutions $x = a \tan t$ and $x = a \sec t$, respectively, over the domains $\left[-\frac{p}{2}, \frac{p}{2}\right]$, and $\left[0, \frac{p}{2}\right]$.

5 Integration by partial fractions

A polynomial P of degree n is a function where the leading coefficient is not zero ($a_n \neq 0$), such that

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = \prod_{k=0}^{n} a_k x^k \qquad (5.1)$$

Let f be a rational function with degree n. We can express f in terms of polynomials P and Q

$$f(x) = \frac{P(x)}{Q(x)} \qquad (5.2)$$

Integration by partial fractions involves decomposing proper rational functions into a sum of simpler rational functions. When encountering an improper fraction, one must use polynomial division to obtain a sum of a polynomial and a proper fraction. That is, if f is an improper fraction (explicitly $\deg(P) \geq \deg(Q)$), through polynomial division in $\mathrm{Z}[x]$ we express f as the following

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \qquad (5.3)$$

Thus obtaining a remainder R, such that $\deg(R) < \deg(Q)$.

We now express $\frac{R(x)}{Q(x)}$ as a sum of partial fractions, obtaining an expression of the form:

$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Ax+b}{(ax^2+bx+c)^n} \qquad (5.4)$$

If Q(x) is a product of distinct linear factors, that is,

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n) = \prod_{k=1}^{n}(a_kx+b_k) \qquad (5.5)$$

Then there exist constants $C_1, C_2 \ldots C_n$, such that:

$$\frac{R(x)}{Q(x)} = \frac{C_1}{(a_1x+b_1)} + \frac{C_2}{(a_2x+b_2)} + \ldots \frac{C_n}{(a_bx+b_b)} = \sum_{k=1}^{n}\frac{C_k}{(a_kx+b_k)} \qquad (5.6)$$

If Q(x) is a product of linear factors, with some not distinct, that is, $(a_i x + b_i)^m$ appears in the factorization, where $m > 1$, then the partial fraction form for the term $\dfrac{C_i}{(a_i x + b_i)}$ would be:

$$\frac{C_i}{(a_i x + b_i)} + \frac{C_i}{(a_i x + b_i)^2} + \ldots \frac{C_i}{(a_i x + b_i)^m} \qquad (5.7)$$

If Q(x) has a factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions for the linear factors of Q(x), there would also be $\dfrac{Cx + B}{ax^2 + bc + c}$, where C and B are constants.

If Q(x) has a factor $(ax^2 + bx + c)^m$ where $m > 1$, then addition to the other partial fractions, there would also be:

$$\frac{C_1 x + B_1}{(ax^2 + bc + c)} + \frac{C_2 x + B_2}{(ax^2 + bc + c)^2} + \ldots + \frac{C_m x + B_m}{(ax^2 + bc + c)^m} \qquad (5.8)$$

Once the proper fraction is rewritten as a sum of partial fractions, each term, or group of terms, can be integrated individually.

Appendix 1: Common integral forms

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int e^x dx = e^x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \sec x dx = \ln\left|\sec x + \tan x\right| + C$$

$$\int \tan x dx = \ln|\sec x| + C$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+1}\right| + C$$

$$\int \frac{1}{x} = \ln|x| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \csc x dx = \ln\left|\csc x - \cot x\right| + C$$

$$\int \cot x dx = \ln|\sin x| + C$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln^{-1}\left|x + \sqrt{x^2 \pm a^2}\right| + C$$

Appendix 2: Important trigonometric identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \cot^2 x = \csc^2 x$$

$$\cos 2x = 1 - \sin^2 x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\sin 2x = 2\sin x \cos x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$